Q1

Friday, October 11, 2013 9:28 AM

When the AWGN has PSD $= \frac{N_0}{2}$, we know that the noise in vector form will have $\sigma^2 = \frac{N_0}{2}$ in each dimension.

To solve part (a) and (b), we first sketch the decision regions. Here, the boundaries of the regions can be easily found because the detector use minimum-distance detection.

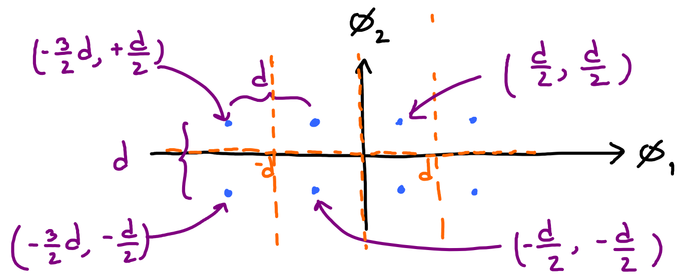
(Earlier, we saw that this detector is optimal (same as the MAP detector) when

(1) the vector channel model is additive Gaussian noise

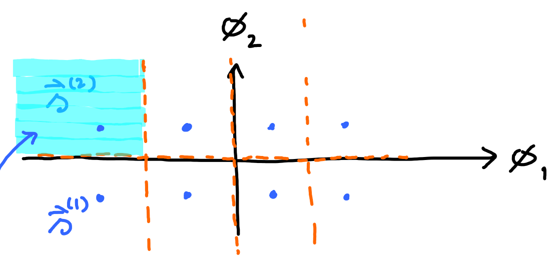
and

(2) the messages are equally likely.)

The boundaries are simply the perpendicular bisectors of the lines connecting two (vertically or horizontally) adjacent signal points.



(a) $Q_{2,2} = P[\hat{W} = 2 \mid W = 1]$



To be detected as $\vec{s}^{(2)}$, the received vector must be in D_2

the decision region for $\vec{s}^{(2)}$.

The first component of $\vec{N}$ (Noise in the 1st dimension)

$$Q_{12} = P\left[-\infty < N_1 \le \frac{d}{2}\right] P\left[\frac{d}{2} \le N_2 < \infty\right]$$

The second component of $\vec{N}$ (Noise in the 2nd dimension)

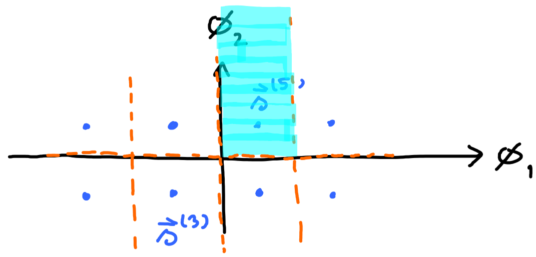
(Noise in the 2^{nd} dimension)

$$= \left(Q\left(\frac{-\infty}{\sigma}\right) - Q\left(\frac{d}{2\sigma}\right)\right)\left(Q\left(\frac{d}{2\sigma}\right) - Q\left(\frac{\infty}{\sigma}\right)\right) = \left(1 - Q\left(\frac{d}{2\sigma}\right)\right)\left(Q\left(\frac{d}{2\sigma}\right)\right)$$

Here, $\sigma^2 = \frac{N_0}{2}$. So $\sigma = \sqrt{\frac{N_0}{2}}$ and $\frac{d}{2\sigma} = \frac{d}{2\times\sqrt{\frac{N_0}{2}}} = \frac{1}{2\sqrt{3}}$.

Therefore, $Q_{12} = \left(1 - Q\left(\frac{1}{2\sqrt{3}}\right)\right)\left(Q\left(\frac{1}{2\sqrt{3}}\right)\right) \approx 0.2371$

(b)



$$Q_{3,5} = P[\hat{W} = 5 \mid W = 3]$$

$$= P\left[\frac{d}{2} \le N_1 \le \frac{3d}{2}\right] P\left[+\frac{d}{2} \le N_2 < \infty\right]$$

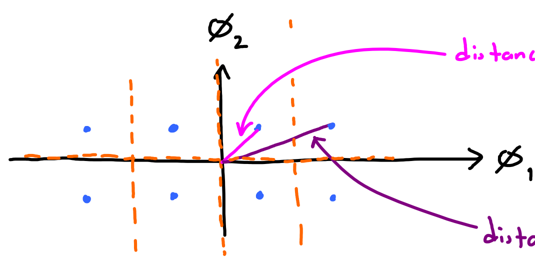
$$= \left(Q\left(\frac{d}{2\sigma}\right) - Q\left(\frac{3d}{2\sigma}\right)\right)\left(Q\left(+\frac{d}{2\sigma}\right) - \cancel{Q\left(\frac{\infty}{\sigma}\right)}^{0}\right)$$

$$= \left(Q\left(\frac{d}{2\sigma}\right) - Q\left(\frac{3d}{2\sigma}\right)\right)\left(\cancel{1-}Q\left(\frac{d}{2\sigma}\right)\right)$$

$$= \left(Q\left(\frac{1}{2\sqrt{3}}\right) - Q\left(\frac{\sqrt{3}}{2}\right)\right)\left(\cancel{1-}Q\left(\frac{1}{2\sqrt{3}}\right)\right)$$

$$\approx 0.0746$$

(c)



distance to origin $= \sqrt{\left(\frac{d}{2}\right)^2 + \left(\frac{d}{2}\right)^2} = \sqrt{\frac{2d^2}{4}} = \frac{d}{\sqrt{2}}$

distance to origin $= \sqrt{\left(\frac{3d}{2}\right)^2 + \left(\frac{d}{2}\right)^2} = \sqrt{\frac{10d^2}{4}} = d\sqrt{\frac{5}{2}}$

$$E_s = \frac{1}{8}\times\left(4\times\left(\frac{d}{\sqrt{2}}\right)^2 + 4\times\left(d\sqrt{\frac{5}{2}}\right)^2\right) = \frac{1}{2}\left(\frac{d^2}{2} + \frac{5d^2}{2}\right) = \frac{1}{2}\times 3d^2 = \frac{3}{2}d^2$$

$$E_b = \frac{E_s}{\log_2 M} = \frac{\frac{3}{2}d^2}{\log_2 8} = \frac{\frac{3}{2}d^2}{3} = \frac{1}{2}d^2 \underset{d=1}{=} \frac{1}{2}$$

$$\frac{E_b}{N_0} = \frac{1/2}{2\times 3} = \frac{1}{12}.$$

Q2

Friday, October 11, 2013 11:43 AM

(a) (i) Step 1

Same arrangement of point vs. decision region.

$$Q = \begin{bmatrix} & 0.3 & 0.3 \\ 0.3 & & 0.3 \\ 0.3 & 0.3 & \end{bmatrix}$$

Step 2

$1-0.3-0.3 = 0.4$

$$Q = \begin{bmatrix} & 0.3 & 0.3 \\ 0.3 & & 0.3 \\ 0.3 & 0.3 & \end{bmatrix}$$

Conclusion: $Q = \begin{bmatrix} 0.4 & 0.3 & 0.3 \\ 0.3 & 0.4 & 0.3 \\ 0.3 & 0.3 & 0.4 \end{bmatrix}$

(ii)

$$Q = \begin{bmatrix} 0.41 & 0.29 & 0.29 & 0.01 \\ 0.29 & 0.41 & 0.29 & 0.01 \\ 0.29 & 0.29 & 0.41 & 0.01 \\ 0.33 & 0.33 & 0.33 & 0.01 \end{bmatrix}$$

(b) $P(\mathcal{E}) = P[\hat{W} \neq W] = 1 - P[\hat{W} = W] = 1 - \sum_i P[\hat{W} = W \mid W = i]\, P[W = i]$

$$= 1 - \sum_i P[\hat{W} = i \mid W = i] \frac{1}{M} = 1 - \sum_i Q_{i,i} \frac{1}{M} = 1 - \frac{1}{M} \operatorname{tr}(Q).$$

equally likely messages

$\operatorname{tr}(A)$ = the sum of the elements on the main diagonal of A.

(i) $P(\mathcal{E}) = 1 - \frac{1}{3}(0.4 + 0.4 + 0.4) = 0.6$

(ii) $P(\mathcal{E}) = 1 - \frac{1}{4}(3 \times 0.41 + 0.01) = 0.69$

Q3

Friday, October 11, 2013 12:57 PM

(a) BPSK waveform channel gives binary symmetric channel with crossover probability $p = Q\left(\frac{d}{2\sigma}\right) = Q\left(\sqrt{\frac{4E_b}{2N_0}}\right) = Q\left(\sqrt{2\frac{E_b}{N_0}}\right)$

$E_s = \left(\frac{d}{2}\right)^2 = \frac{d^2}{4}$

$E_b = \frac{E_s}{\log_2 M} = \frac{d^2/4}{\log_2 2} = \frac{d^2}{4}$

$d = \sqrt{4E_b}$

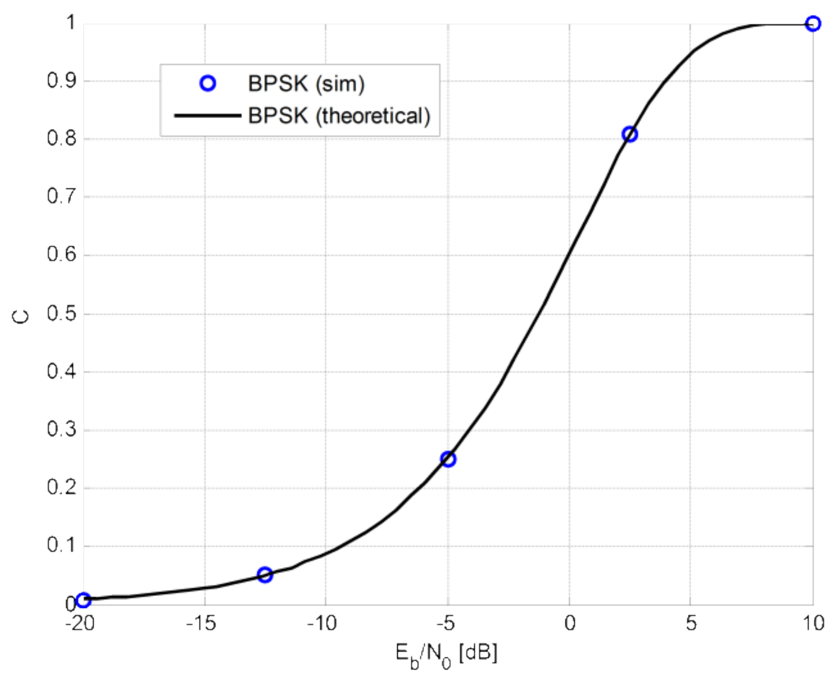
$\sigma^2 = \frac{N_0}{2}$

$\sigma = \sqrt{\frac{N_0}{2}}$

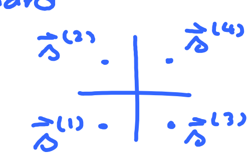
$2\sigma = \sqrt{2N_0}$

The capacity of BSC is given by $1 - H(p) =$

$= 1 + \left(p\log_2 p + (1-p)\log_2(1-p)\right)$ where $p = Q\left(\sqrt{2\frac{E_b}{N_0}}\right)$.



(b) In class, we have shown that the Q matrix of standard rectangular quaternary QAM is given by

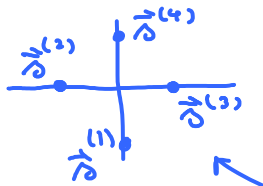


$$Q = \begin{bmatrix} (1-q)^2 & q(1-q) & q(1-q) & q^2 \\ q(1-q) & (1-q)^2 & q^2 & q(1-q) \\ q(1-q) & q^2 & (1-q)^2 & q(1-q) \\ q^2 & q(1-q) & q(1-q) & (1-q)^2 \end{bmatrix}$$ where $q = Q\left(\frac{d}{2\sigma}\right)$

This is a symmetric channel and the corresponding capacity is

$$C = \log_2 |S_Y| - H(\vec{r}) = \log_2 4 - H\left(\left[(1-q)^2 \quad q(1-q) \quad q(1-q) \quad q^2\right]\right)$$

To get the constellation for QPSK, we simply rotate the constellation above by 45°.



Because the relative positions of the points in the constellation are the same, we have the same Q matrix (if the points are numbered as shown)

So, the capacity is the same as what we found above.

Note also that $E_s = \left(\frac{d}{\sqrt{2}}\right)^2 = \frac{d^2}{2}$ and $E_b = \frac{E_s}{\log_2 M} = \frac{E_s}{\log_2 4} = \frac{d^2/2}{2} = \frac{d^2}{4}$

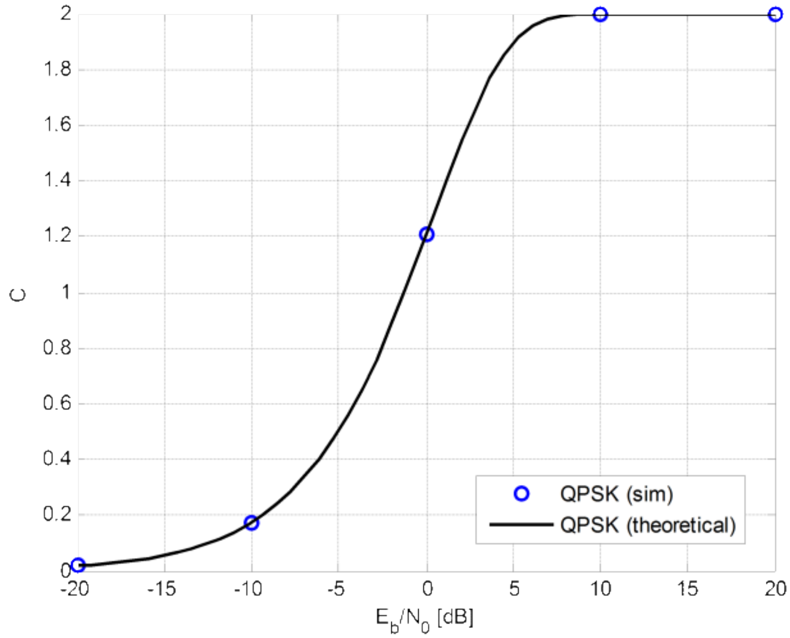
$$d = \sqrt{4 E_b}$$

$$\frac{d}{2\sigma} = \sqrt{\frac{4 E_b}{2 N_0}} = \sqrt{2 \frac{E_b}{N_0}}.$$

Therefore,

$$C = 2 + (1-q)^2 \log_2 (1-q)^2 + 2q(1-q) \log_2 q(1-q) + q^2 \log_2 q^2$$

where $q = Q\left(\sqrt{2 \frac{E_b}{N_0}}\right)$


C
E_b/N_0 [dB]
QPSK (sim)
QPSK (theoretical)

Q4

Friday, October 11, 2013 1:35 PM

$$G = \left[\begin{array}{cccc:cc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{array}\right]$$

$$\underbrace{\begin{matrix}1&0&0&0\\0&1&1&1\end{matrix}}_{P} \qquad \underbrace{\begin{matrix}1&0\\0&1\end{matrix}}_{I}$$

(a) Any codeword is of the form $\underline{x} = \underline{b}\,G$.

Here, G has $k = 2$ rows; so, the length of $\underline{b}$ must also be $k = 2$.

There fore, there are $2^k = 2^2 = 4$ codewords.

$\underline{b}$	$\underline{x} = \underline{b}\,G$
0 0	0 0 0 0 0 0
0 1	0 1 1 1 0 1
1 0	1 0 0 0 1 0
1 1	1 1 1 1 1 1

(b) $H = [I \,\vdots\, -P^T] = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}$

the negative sign does not matter in GF(2)

(c)

add these two rows

$$GH^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \checkmark$$

(d) $d_{min} = 2 \leftarrow$ There are $\binom{4}{2} = 6$ pairs of codewords.

The minimum Hamming distance among these pairs is 2.

Q5

Friday, October 11, 2013 1:48 PM

$$H = \left[\begin{array}{ccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{array}\right]$$

I $\qquad$ P

(a) Matrix H should be $(n-k) \times n$.

So, $n = 7$,

$n - k = 3$,

$k = n - 3 = 7 - 3 = 4$.

(b)

$$G = \left[-P^T \mid I \right] = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

the negative sign does not matter in $GF(2)$

(c)

(i) From (a), we know that $k = 4$. Therefore, the information bits should be divided into blocks of 4 bits.
12 bits are given. So, there should be $\frac{12}{4} = 3$ blocks.

(ii)

$\underline{b}$	$\underline{x} = \underline{b}\, G$
0 0 1 1	0 0 1 0 0 1 1
1 0 1 1	0 1 0 1 0 1 1
1 0 1 0	1 0 1 1 0 1 0

(d)

(i) From (a), we know that $n = 7$. Therefore, the received bits should be divided into blocks of 7 bits
21 bits are given. So, there should be $\frac{21}{7} = 3$ blocks.

(ii)

$\underline{y}$	$\underline{s} = \underline{y} H^T$		Error position	Corrected $\underline{x}$
1 0 1 1 0 1 1	1 1 1	→ 7^{th} column of H	7	1 0 1 1 0 1 0
1 0 0 0 0 0 1	0 1 1	→ 4^{th} column of H	4	1 0 0 1 0 0 1
0 1 0 0 0 0 1	1 0 1	→ 5^{th} column of H	5	0 1 0 $\underbrace{0\ 1\ 0\ 1}_{\hat{\underline{b}}}$